

Upper Bound For The Ratios Of Eigenvalues Of Schrödinger Operators With Nonnegative Single-Barrier Potentials

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Abstract

In this paper we prove the optimal upper bound $\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2} \left(\lambda_n > \lambda_m \geq 11 \sup_{x \in [0,1]} q(x) \right)$ for one-dimensional Schrödinger operators with a nonnegative differentiable and single-barrier potential $q(x)$, such that $|q'(x)| \leq q^*$, where $q^* = \frac{2}{15} \inf\{q(0), q(1)\}$. In particular, if $q(x)$ satisfies the additional condition $\sup_{x \in [0,1]} q(x) \leq \frac{\pi^2}{11}$, then $\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}$ for $n > m \geq 1$. For this result, we develop a new approach to study the monotonicity of the modified Prüfer angle function.

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1 Introduction

Consider the Sturm-Liouville equation acting on $[0, 1]$

$$-y'' + q(x)y = \lambda y, \quad (1.1)$$

with Dirichlet boundary conditions

$$y(0) = y(1) = 0. \quad (1.2)$$

Here q is a nonnegative differentiable and single-barrier potential in $[0, 1]$.

It is known [3], that f is a single-barrier (resp. single-well) function on $[0, 1]$ if there is a point $x_0 \in [0, 1]$ such that f is increasing (resp. decreasing) on $[0, x_0]$ and decreasing (resp.

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increasing) on $[x_0, 1]$.

It is known (see [7]) that the spectrum of Problem (1.1) – (1.2) consists of a growing sequence of infinitely point $\lambda_1 < \lambda_2 < \dots < \lambda_n \dots \infty$.

Ashbaugh and Benguria [1, 2] proved the optimal bound $\frac{\lambda_n}{\lambda_1} \leq n^2$, for nonnegative potentials.

They also established the ratio estimate $\frac{\lambda_n}{\lambda_m} \leq (\lceil \frac{n}{m} \rceil)^2$, for $n > m \geq 1$, where $\lceil s \rceil$ denotes the smallest integer greater than or equal to s . Later, Yu-Ling Huang and C. K. Law [5] extended the results in [2] to more general boundary conditions. The same authors in [6] proved that the eigenvalues of the regular Sturm-Liouville equation $-(p(x)y')' + q(x)y = \lambda \rho(x)y$ (with Dirichlet boundary conditions) satisfy the lower bound

$$\frac{\lambda_n}{\lambda_m} \geq \frac{1}{1+\xi} \left(\frac{n+1}{m+1} \right)^2 \frac{k}{K}, \quad n > m \geq 0,$$

where $q(x) \geq 0$ and $0 < k \leq p\rho(x) \leq K$, $\xi = \frac{K \max\{pq\}}{kn^2\sigma^2\pi^2}$ and $\sigma = \left(\int_0^1 \frac{1}{p(s)} ds \right)^{-1}$. In 2005, M. Horváth and M. Kiss [4] showed that

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}, \quad n > m \geq 1, \quad (1.3)$$

for nonnegative single-well potentials. Their approach is mainly based on the monotonicity of the Prüfer angle as function in $\lambda \geq 0$ (see [4, Theorem 2.2]). At the end of their paper [4, Remark 5.1], they gave an example of a single-barrier potential which shows that the associated Prüfer angle is not a monotonous function.

In the present paper, we give additional conditions on the single-barrier potential $q(x)$ for which Theorem 2.2 in [4] and the estimate (1.3) remain valid. Namely, we prove that if $q(x)$ is a nonnegative and single-barrier potential (with transition point $x_0 \in [0, 1]$), such that $|q'(x)| \leq q^*$ where $q^* = \frac{2}{15} \inf\{q(0), q(1)\}$, then $\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}$ for $\lambda_n > \lambda_m \geq 11q(x_0)$. In particular, if $q(x)$ satisfies the additional condition $q(x_0) \leq \frac{\pi^2}{11}$, then the last bound estimate holds for $n > m \geq 1$. Note that, our approach used in this paper can be applied to the case of nonnegative and single-well potentials studied in [4] (without further restrictions on $q(x)$).

2 Preliminaries And The Main Statement

Following [4], we introduce the modified Prüfer transformation.

Denote by $y(x, z)$ the unique solution of the initial value problem

$$-y'' + q(x)y = z^2y, \quad x \in [0, 1], \quad z > 0, \quad (2.1)$$

$$y(0) = 0, \quad y'(0) = 1. \quad (2.2)$$

The Prüfer variables $r(x, z)$, $\varphi(x, z)$ that we use here are defined by

$$y(x, z) = r(x, z) \sin \varphi(x, z), \quad (2.3)$$

$$y'(x, z) = zr(x, z) \cos \varphi(x, z), \quad (2.4)$$

$$\varphi(0, z) = 0, \quad (2.5)$$

where $r(x, z) > 0$, and then let $\theta(x, z) = \frac{\varphi(x, z)}{z}$.

We denote by prime (resp. dot) the derivative with respect to x (resp. z).

Using Equation (1.1) one finds the following differential equations for $r(x, z)$ and $\varphi(x, z)$:

$$\varphi' = z - \frac{q}{z} \sin^2 \varphi, \quad (2.6)$$

and

$$\frac{r'}{r} = \frac{q}{z} \sin \varphi \cos \varphi. \quad (2.7)$$

It is obvious that z^2 is an eigenvalue iff $\varphi(\pi, z)$ is a multiple of π . Denote by z_n the square root of λ_n ($\lambda_n = z_n^2$). Moreover by (2.6), $\varphi'(x, z) > 0$ for $x \in [0, x_0]$ and $z^2 > q(x_0)$. In this case φ^{-1} exists and $\varphi^{-1}(k\pi + \frac{\pi}{2})$, $\varphi^{-1}((k+1)\pi)$ ($k \in \mathbb{N}$) are the zeros of y' and y in $(0, x_0]$, respectively. It is known (e.g., see [7, chap.1]) that these zeros are decreasing as z increases. We now enunciate the main result of this paper.

Theorem 2.1. *For the Sturm-Liouville problem (1.1)-(1.2), if $q(x)$ is a nonnegative differentiable and single-barrier potential with transition point $x_0 \in [0, 1]$ such that $|q'(x)| \leq q^*$ where $q^* = \frac{2}{15} \inf\{q(0), q(1)\}$, then*

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}, \quad \text{for } \lambda_n > \lambda_m \geq 11q(x_0). \quad (2.8)$$

In particular, if $q(x)$ satisfies the additional condition $q(x_0) \leq \frac{\pi^2}{11}$, then $\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}$, for $n > m \geq 1$.

If for two different m and n equality holds, then $q \equiv 0$ in $[0, 1]$.

The proof of Theorem 2.1 will be given in Section 4.

3 Monotonicity Of The Prüfer Angle Function $\theta(x_0, z)$.

In this section, we study the monotonicity of the Prüfer angle function $\theta(x_0, z)$.

Theorem 3.1. *Let $q(x) \geq 0$ be monotone increasing and differentiable in $[0, x_0]$ such that $q'(x) \leq \frac{2}{15}q(0)$. Then $\dot{\theta}(x_0, z) \geq 0$ for $z \geq \sqrt{11q(x_0)}$. If there is a $z \geq \sqrt{11q(x_0)}$ with $\dot{\theta}(x_0, z) = 0$, then $q \equiv 0$ in $[0, x_0]$.*

To prove this theorem we need some preliminary results.

Lemma 3.1. *(Corollary 3.3 in [4])*

$$\dot{\theta}(x, z) = \frac{2}{z^2 r^2(x)} \int_0^x r^2(t) \frac{q(t)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dt. \quad (3.1)$$

The following result plays a fundamental role in the sequel.

Lemma 3.2. *Let $i \geq 0$ be an integer and assume that $\varphi^{-1}(i\pi + \frac{\pi}{2} + D) \in (0, x_0]$, where $0 \leq D \leq \pi$. Then*

$$\begin{aligned} & \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} r^2(x) \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \\ & \geq r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left[\int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \right. \\ & \quad \left. - 2 \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin \varphi \cos \varphi \left(\int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \right]. \end{aligned} \quad (3.2)$$

Proof. By (2.7),

$$\begin{aligned} & - \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} r^2(x) \frac{q(x)}{z} \varphi \sin \varphi \cos \varphi dx \\ & = - \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} r(x) r'(x) \varphi dx \\ & = - \frac{1}{2} \left[r^2(x) \varphi(x) \right]_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} + \frac{1}{2} \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} r^2(x) \varphi'(x) dx. \end{aligned}$$

Since $r^2(x) = e^{2 \int_0^x \frac{q(s)}{z} \sin \varphi \cos \varphi ds}$, then

$$\begin{aligned} & - \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} r^2(x) \frac{q(x)}{z} \varphi \sin \varphi \cos \varphi dx \\ & = - \frac{1}{2} \left[\left(i\pi + \frac{\pi}{2} + D \right) r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) - \left(i\pi + \frac{\pi}{2} \right) r^2(\varphi^{-1}(i\pi + \frac{\pi}{2})) \right] \\ & \quad + \frac{1}{2} \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} r^2(x) \varphi'(x) dx \\ & = - \frac{1}{2} r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left[D - \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{r^2(x)}{r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))} \varphi'(x) dx \right] \\ & \quad - \left(\frac{i\pi + \frac{\pi}{2}}{2} \right) r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left[1 - \frac{r^2(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))} \right] \\ & = - \frac{1}{2} r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left[D - \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \varphi'(x) g(x, z) dx \right] \\ & \quad - \left(\frac{i\pi + \frac{\pi}{2}}{2} \right) r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left[1 - g(\varphi^{-1}(i\pi + \frac{\pi}{2}), z) \right], \end{aligned} \quad (3.3)$$

where $g(x, z) = e^{-2 \int_x^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(s)}{z} \sin \varphi \cos \varphi ds}$.

Using the inequality

$$\begin{aligned} g(x, z) & = e^{-2 \int_x^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(s)}{z} \sin \varphi \cos \varphi ds} \\ & \geq 1 - 2 \int_x^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(s)}{z} \sin \varphi \cos \varphi ds, \end{aligned}$$

we obtain

$$\begin{aligned}
& -\frac{1}{2}r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left[D - \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \varphi'(x)g(x, z)dx \right] \\
& \geq -r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \varphi'(x) \left(\int_x^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(s)}{z} \sin \varphi \cos \varphi ds \right) dx \\
& = r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left(- \left[\varphi(x) \int_x^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(s)}{z} \sin \varphi \cos \varphi ds \right]_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \right. \\
& \quad \left. - \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \varphi(x) \sin \varphi \cos \varphi dx \right) \\
& = r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left((i\pi + \frac{\pi}{2}) \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin \varphi \cos \varphi dx \right. \\
& \quad \left. - \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \varphi(x) \sin \varphi \cos \varphi dx \right). \tag{3.4}
\end{aligned}$$

Similarly,

$$\begin{aligned}
& -(\frac{i\pi + \frac{\pi}{2}}{2})r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left[1 - g(\varphi^{-1}(i\pi + \frac{\pi}{2}), z) \right] \\
& \geq (\frac{i\pi + \frac{\pi}{2}}{2})r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin \varphi \cos \varphi dx. \tag{3.5}
\end{aligned}$$

Using (3.4) and (3.5),

$$\begin{aligned}
& - \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} r^2(x) \frac{q(x)}{z} \varphi \sin \varphi \cos \varphi dx \\
& \geq -r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \varphi(x) \sin \varphi \cos \varphi dx. \tag{3.6}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} r^2(x) \frac{q(x)}{z} \sin^2 \varphi dx \\
& = r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin^2 \varphi(x) g(x, z) dx \\
& \geq r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left[\int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin^2 \varphi dx \right. \\
& \quad \left. - 2 \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin^2 \varphi \left(\int_x^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(s)}{z} \sin \varphi \cos \varphi ds \right) dx \right].
\end{aligned}$$

Integrating by parts, yields

$$\begin{aligned}
& -2 \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin^2 \varphi \left(\int_x^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(s)}{z} \sin \varphi \cos \varphi ds \right) dx \\
& = -2 \left[\int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \int_x^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(s)}{z} \sin \varphi \cos \varphi ds \right]_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)}
\end{aligned}$$

$$\begin{aligned}
& -2 \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \sin \varphi \cos \varphi \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \\
& = -2 \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \sin \varphi \cos \varphi \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} r^2(x) \frac{q(x)}{z} \sin^2 \varphi dx \\
& \geq r^2(\varphi^{-1}(i\pi + \frac{\pi}{2} + D)) \left[\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \sin^2 \varphi dx \right. \\
& \quad \left. - 2 \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \sin \varphi \cos \varphi \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \right]. \tag{3.7}
\end{aligned}$$

From (3.6) and (3.7), we get (3.2). This completes the proof of the lemma. \square

Using the substitution $t = \varphi(x) - (i+1)\pi$ ($i \geq 0$), we obtain the following result:

Lemma 3.3. Assume that $\varphi^{-1}(i\pi + \frac{\pi}{2} + D) \in (0, x_0]$ ($i \geq 0$, $0 \leq D \leq \pi$), then

$$\begin{aligned}
& \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \\
& = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \frac{\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \left(\sin^2 t - t \sin t \cos t \right)}{1 - \frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t} dt \\
& \quad - (i+1)\pi \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \frac{\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin t \cos t}{1 - \frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t} dt. \tag{3.8}
\end{aligned}$$

Lemma 3.4. Let $q(x) \geq 0$ be monotone increasing on $[0, x_0]$ such that $z^2 > q(x_0)$ and $\varphi^{-1}(i\pi + \frac{\pi}{2} + D) \in (0, x_0]$ ($i \geq 0$) with $\frac{\pi}{2} \leq D \leq \pi$. Then

i)

$$\frac{\pi}{4} \left(\frac{\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2}}{1 - \frac{3q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{4z^2}} \right) \leq \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \sin^2 \varphi dx \leq \frac{\pi}{2} \left(\frac{\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}+D))}{z^2}}{1 - \frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}+D))}{z^2}} \right), \tag{3.9}$$

ii)

$$\begin{aligned}
& \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \frac{\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \left(\sin^2 t - t \sin t \cos t \right)}{1 - \frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t} dt \\
& \geq \frac{\pi}{4} \left(\frac{\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2}}{1 - \frac{3q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{4z^2}} \right) + \frac{\pi}{4} \log \left(1 - \frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2} \right) \\
& \quad - \frac{\pi}{6} \log \left(1 - \frac{3q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{4z^2} \right), \tag{3.10}
\end{aligned}$$

iii) there exists $c_1 \in (i\pi + \frac{\pi}{2}, i\pi + \frac{\pi}{2} + D)$ ($i \geq 0$) such that

$$\begin{aligned} & -(i+1)\pi \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \frac{\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin t \cos t}{1 - \frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t} dt \\ & \geq -\frac{(i+1)\pi^2}{2z^2} \left(\frac{1}{1 - \frac{q(\varphi^{-1}(c_1))}{z^2}} \right) \frac{q'(\varphi^{-1}(c_1))}{\varphi'(\varphi^{-1}(c_1))}. \end{aligned} \quad (3.11)$$

Proof. i) By virtue of Lemma 3.3,

$$\int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin^2 \varphi dx = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \frac{\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t}{1 - \frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t} dt.$$

For $z^2 > q(x_0)$, $|\frac{q(\varphi^{-1}(t))}{z^2} \sin^2(t)| < 1$, and hence, the function $\frac{1}{1 - \frac{q}{z^2} \sin^2 \varphi}$ is developable in entire series. Thus, since $q(x) \geq 0$ is monotone increasing, then

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \frac{\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t}{1 - \frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t} dt \\ & = \sum_{n \geq 0} \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \left(\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \right)^{n+1} (\sin t)^{2n+2} dt \\ & \geq \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} (\sin t)^{2n+2} dt. \end{aligned} \quad (3.12)$$

By integration by parts, we obtain

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} (\sin t)^{2n+2} dt \geq \int_{-\frac{\pi}{2}}^0 (\sin t)^{2n+2} dt = \int_{-\frac{\pi}{2}}^0 (\sin t)^{2n+1} \sin(t) dt \\ & = (2n+1) \int_{-\frac{\pi}{2}}^0 (\sin t)^{2n} \cos^2(t) dt \\ & = (2n+1) \int_{-\frac{\pi}{2}}^0 (\sin t)^{2n} dt - (2n+1) \int_{-\frac{\pi}{2}}^0 (\sin t)^{2n+2} dt. \end{aligned}$$

Therefore,

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} (\sin t)^{2n+2} dt \geq \frac{2n+1}{2n+2} \int_{-\frac{\pi}{2}}^0 (\sin t)^{2n} dt \\ & = \prod_{p=1}^n \frac{2p+1}{2p+2} \int_{-\frac{\pi}{2}}^0 \sin^2(t) dt = \frac{\pi}{4} \prod_{p=1}^n \frac{2p+1}{2p+2} \geq \frac{\pi}{4} \left(\frac{3}{4} \right)^n. \end{aligned} \quad (3.13)$$

Then

$$\begin{aligned} & \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin^2 \varphi dx \\ & \geq \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \int_{-\frac{\pi}{2}}^0 (\sin t)^{2n+2} dt \end{aligned}$$

$$\begin{aligned}
&\geq \frac{\pi}{4} \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \sum_{n \geq 0} \left(\frac{3q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{4z^2} \right)^n \\
&= \frac{\pi}{4} \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{3q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{4z^2}} \right). \tag{3.14}
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \frac{\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t}{1 - \frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t} dt \\
&\leq \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2} \right)^{n+1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^{2n+2} dt \\
&\leq \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2} \right)^{n+1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(t) dt \\
&= \frac{\pi}{2} \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2}}{1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2}} \right). \tag{3.15}
\end{aligned}$$

Using (3.14) and (3.15), we find (3.9).

ii) Clearly, if $|t| \in]0, \frac{\pi}{2}[$, then $\sin^2 t - t \sin t \cos t > 0$, and hence,

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \frac{\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} (\sin^2 t - t \sin t \cos t)}{1 - \frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2(t)} dt \\
&\geq \int_{-\frac{\pi}{2}}^0 \frac{\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} (\sin^2 t - t \sin t \cos t)}{1 - \frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2(t)} dt \\
&= \sum_{n \geq 0} \int_{-\frac{\pi}{2}}^0 \left(\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \right)^{n+1} (\sin t)^{2n} (\sin^2 t - t \sin t \cos t) dt \\
&\geq \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \int_{-\frac{\pi}{2}}^0 (\sin t)^{2n+2} dt \\
&\quad - \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \int_{-\frac{\pi}{2}}^0 t (\sin t)^{2n+1} \cos(t) dt.
\end{aligned}$$

Integrating by parts and using (3.13), yields

$$\begin{aligned}
& - \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \int_{-\frac{\pi}{2}}^0 t (\sin t)^{2n+1} \cos t dt \\
&= - \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \left[\frac{t (\sin t)^{2n+2}}{2n+2} \right]_{-\frac{\pi}{2}}^0 \\
&\quad + \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \frac{1}{2n+2} \int_{-\frac{\pi}{2}}^0 (\sin t)^{2n+2} dt \\
&\geq - \frac{\pi}{4} \sum_{n \geq 0} \frac{1}{n+1} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} + \frac{\pi}{8} \sum_{n \geq 0} \frac{1}{n+1} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \left(\frac{3}{4} \right)^n
\end{aligned}$$

$$= \frac{\pi}{4} \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) - \frac{\pi}{6} \log \left(1 - \frac{3q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{4z^2} \right). \quad (3.16)$$

Therefore, (3.14) and (3.16) give (3.10).

iii) As before,

$$\begin{aligned} & - \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \frac{\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin t \cos t}{1 - \frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \sin^2 t} dt \\ &= - \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+D} \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(t+(i+1)\pi))}{z^2} \right)^{n+1} \sin^{2n+1}(t) \cos(t) dt \\ &\geq - \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \int_{-\frac{\pi}{2}}^0 \sin^{2n+1}(t) \cos(t) dt \\ &\quad - \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}((i+1)\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \int_0^{-\frac{\pi}{2}+D} \sin^{2n+1}(t) \cos(t) dt \\ &= \frac{1}{2} \left[\sum_{n \geq 0} \frac{1}{n+1} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} - \sum_{n \geq 0} \frac{1}{n+1} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2} \right)^{n+1} \right] \\ &= \frac{1}{2} \left[\log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2} \right) - \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) \right]. \quad (3.17) \end{aligned}$$

By the mean value theorem, there exists $c_1 \in (i\pi + \frac{\pi}{2}, i\pi + \frac{\pi}{2} + D)$ such that

$$\begin{aligned} & \left[\log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2} \right) - \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) \right] \\ &= -\frac{D}{z^2} \left(\frac{1}{1 - \frac{q(\varphi^{-1}(c_1))}{z^2}} \right) \frac{\partial q(\varphi^{-1}(c_1))}{\partial t} \\ &\geq -\frac{\pi}{z^2} \left(\frac{1}{1 - \frac{q(\varphi^{-1}(c_1))}{z^2}} \right) \frac{\partial q(\varphi^{-1}(c_1))}{\partial t} \\ &= -\frac{\pi}{z^2} \left(\frac{1}{1 - \frac{q(\varphi^{-1}(c_1))}{z^2}} \right) \frac{q'(\varphi^{-1}(c_1))}{\varphi'(\varphi^{-1}(c_1))}. \end{aligned}$$

Therefore, from this and (3.17), we obtain (3.11). □

Lemma 3.5. *Let $q(x)$ be satisfying the conditions in Lemma 3.4. Then there exists $c_2 \in (i\pi + \frac{\pi}{2}, i\pi + \frac{\pi}{2} + D)$ ($\frac{\pi}{2} \leq D \leq \pi$) such that*

$$\begin{aligned} & -2 \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin \varphi \cos \varphi \left(\int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \\ &\geq -\frac{\pi}{4} \left[\left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{3}{4} \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}} \right) \log \left(1 - \frac{3}{4} \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) - 2 \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}} \right) \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) \right] \\ &\quad + \frac{\pi^2}{2z^2} \left(\frac{\log(1 - \frac{q(\varphi^{-1}(c_2))}{z^2}) - \frac{q(\varphi^{-1}(c_2))}{z^2}}{(1 - \frac{q(\varphi^{-1}(c_2))}{z^2})^2} \right) \frac{q'(\varphi^{-1}(c_2))}{\varphi'(\varphi^{-1}(c_2))}. \quad (3.18) \end{aligned}$$

Proof. For $z^2 > q(x_0)$, we have

$$-2 \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin \varphi \cos \varphi \left(\int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx$$

$$\begin{aligned}
&= -2 \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{\frac{q(x)}{z} \varphi' \sin \varphi \cos \varphi}{z - \frac{q(x)}{z} \sin^2 \varphi} \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \\
&= -2 \sum_{n \geq 0} \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \left(\frac{q(x)}{z^2} \right)^{n+1} \varphi' \sin^{2n+1}(\varphi) \cos(\varphi) \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx.
\end{aligned}$$

Integrating by parts and taking into account that $q(x)$ be monotone increasing on $[0, x_0]$, we find

$$\begin{aligned}
&-2 \sum_{n \geq 0} \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \left(\frac{q(x)}{z^2} \right)^{n+1} \varphi' \sin^{2n+1}(\varphi) \cos(\varphi) \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \\
&= -2 \sum_{n \geq 0} \left[\frac{\sin^{2n+2}(\varphi)}{2n+2} \left(\frac{q(x)}{z^2} \right)^{n+1} \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right]_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \\
&+ \sum_{n \geq 0} \frac{1}{n+1} \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \sin^{2n+4}(\varphi) \left(\frac{q(x)}{z^2} \right)^{n+1} \frac{q(x)}{z} dx \\
&+ \sum_{n \geq 0} \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \left(\frac{q(x)}{z^2} \right)^n \frac{q'(x)}{z^2} \sin^{2n+2}(\varphi) \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \\
&\geq - \sum_{n \geq 0} \left(\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}+D))}{z^2} \right)^{n+1} \frac{1}{n+1} \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(s)}{z} \sin^2(\varphi) ds \right) \\
&+ \sum_{n \geq 0} \frac{1}{n+1} \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \left(\frac{q(x)}{z^2} \right)^{n+1} \frac{q(x)}{z} \sin^{2n+4}(\varphi) dx \\
&\geq - \left[\sum_{n \geq 0} \frac{1}{n+1} \left(\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}+D))}{z^2} \right)^{n+1} \right] \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \sin^2 \varphi dx \\
&+ \sum_{n \geq 0} \frac{1}{n+1} \left(\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2} \right)^{n+1} \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \sin^{2n+4} \varphi dx.
\end{aligned}$$

By (3.9),

$$\begin{aligned}
&- \left[\sum_{n \geq 0} \frac{1}{n+1} \left(\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}+D))}{z^2} \right)^{n+1} \right] \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \sin^2 \varphi dx \\
&\geq \frac{\pi}{2} \left(\frac{\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}+D))}{z^2}}{1 - \frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}+D))}{z^2}} \right) \log \left(1 - \frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}+D))}{z^2} \right). \tag{3.19}
\end{aligned}$$

On the other hand, using (3.13),

$$\begin{aligned}
&\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \sin^{2n+4} \varphi dx \\
&= \sum_{p \geq 0} \int_{i\pi+\frac{\pi}{2}}^{i\pi+\frac{\pi}{2}+D} \left(\frac{q(\varphi^{-1}(t))}{z^2} \right)^{p+1} \sin^{2(n+p)+4}(t) dt \\
&\geq \sum_{p \geq 0} \left(\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2} \right)^{p+1} \int_{i\pi+\frac{\pi}{2}}^{i\pi+\frac{\pi}{2}+D} \sin^{2(n+p+1)+2}(t) dt \\
&\geq \frac{\pi}{4} \left(\frac{3}{4} \right)^{n+1} \frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2} \sum_{p \geq 0} \left(\frac{3q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{4z^2} \right)^p
\end{aligned}$$

$$\geq \frac{\pi}{4} \left(\frac{3}{4} \right)^{n+1} \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{3}{4} \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}} \right).$$

Thus

$$\begin{aligned} & \sum_{n \geq 0} \frac{1}{n+1} \left(\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right)^{n+1} \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin^{2n+4} \varphi dx \\ & \geq -\frac{\pi}{4} \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{3}{4} \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}} \right) \log \left(1 - \frac{3}{4} \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right). \end{aligned} \quad (3.20)$$

Therefore, by (3.19) and (3.20),

$$\begin{aligned} & -2 \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \sin \varphi \cos \varphi \left(\int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \\ & \geq \frac{\pi}{4} \left[2 \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2}}{1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2}} \right) \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2} \right) - \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{3}{4} \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}} \right) \log \left(1 - \frac{3}{4} \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) \right] \\ & = \frac{\pi}{2} \left[\left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2}}{1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2}} \right) \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2} \right) - \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}} \right) \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) \right] \\ & - \frac{\pi}{4} \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{3}{4} \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}} \right) \log \left(1 - \frac{3}{4} \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) \\ & - \frac{\pi}{2} \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}} \right) \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right). \end{aligned} \quad (3.21)$$

By the mean value theorem, there exists $c_2 \in (i\pi + \frac{\pi}{2}, i\pi + \frac{\pi}{2} + D)$ such that

$$\begin{aligned} & \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2}}{1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2}} \right) \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2} + D))}{z^2} \right) - \left(\frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}} \right) \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) \\ & = \frac{D}{z^2} \left(\frac{\log(1 - \frac{q(\varphi^{-1}(c_2))}{z^2}) - \frac{q(\varphi^{-1}(c_2))}{z^2}}{(1 - \frac{q(\varphi^{-1}(c_2))}{z^2})^2} \right) \frac{\partial q(\varphi^{-1}(c_2))}{\partial t} \\ & \geq \frac{\pi}{z^2} \left(\frac{\log(1 - \frac{q(\varphi^{-1}(c_2))}{z^2}) - \frac{q(\varphi^{-1}(c_2))}{z^2}}{(1 - \frac{q(\varphi^{-1}(c_2))}{z^2})^2} \right) \frac{\partial q(\varphi^{-1}(c_2))}{\partial t} \\ & = \frac{\pi}{z^2} \left(\frac{\log(1 - \frac{q(\varphi^{-1}(c_2))}{z^2}) - \frac{q(\varphi^{-1}(c_2))}{z^2}}{(1 - \frac{q(\varphi^{-1}(c_2))}{z^2})^2} \right) \frac{q'(\varphi^{-1}(c_2))}{\varphi'(\varphi^{-1}(c_2))}. \end{aligned} \quad (3.22)$$

From this and (3.21), we get (3.18). \square

We are now ready to prove Theorem 3.1.

Proof. If for some $\lambda > 0$, $y'(x, z)$ has no zeros in $(0, 1)$, then $\varphi(x, z) < \frac{\pi}{2}$ for $x \in (0, x_0)$, and hence

$$\int_0^{x_0} r^2(x) \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \geq 0. \quad (3.23)$$

In the rest of the proof let $\varphi(x_0, z) = k\pi + \frac{\pi}{2} + D$, with $k \geq 0$ be an integer and $0 \leq D \leq \pi$. Then

$$\begin{aligned}
& \int_0^{x_0} r^2(x) \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \\
&= \int_0^{\varphi^{-1}(k\pi + \frac{\pi}{2} + D)} r^2(x) \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \\
&= \int_0^{\varphi^{-1}(\frac{\pi}{2})} \tau(x) dx + \sum_{i=0}^{k-1} \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}((i+1)\pi + \frac{\pi}{2})} \tau(x) dx + \int_{\varphi^{-1}(k\pi + \frac{\pi}{2})}^{\varphi^{-1}(k\pi + \frac{\pi}{2} + D)} \tau(x) dx, \quad (3.24)
\end{aligned}$$

where $\tau(x) = r^2(x) \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right)$. Since $\varphi \in]0, \frac{\pi}{2}]$, then

$$\int_0^{\varphi^{-1}(\frac{\pi}{2})} r^2(x) \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \geq 0. \quad (3.25)$$

It is easily seen that if $0 \leq D \leq \frac{\pi}{2}$, then

$$\int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} r^2(x) \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \geq 0. \quad (3.26)$$

If $\frac{\pi}{2} \leq D \leq \pi$, then in view of Lemma 3.3, together with (3.10) and (3.11),

$$\begin{aligned}
& \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{\pi}{2} + D)} \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \\
& \geq \frac{\pi}{4} \frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{3q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{4z^2}} + \frac{\pi}{4} \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) - \frac{\pi}{6} \log \left(1 - \frac{3q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{4z^2} \right) \\
& \quad - \frac{(i+1)\pi^2}{2z^2} \left(\frac{1}{1 - \frac{q(\varphi^{-1}(c_1))}{z^2}} \right) \frac{q'(\varphi^{-1}(c_1))}{\varphi'(\varphi^{-1}(c_1))}. \quad (3.27)
\end{aligned}$$

Under the hypotheses $z^2 \geq 11q(x_0)$, $\varphi'(\varphi^{-1}(c_1)) \geq \frac{10z}{11}$ and $\frac{1}{1 - \frac{q(\varphi^{-1}(c_1))}{z^2}} \leq \frac{11}{10}$. On the other hand, recall that if $x_0 = \varphi^{-1}(i\pi + \frac{\pi}{2} + D)$ ($\frac{\pi}{2} \leq D \leq \pi$), then the solution $y(x, z)$ of (1.1)-(1.2) has exactly $(i+1)$ zeros in $(0, x_0]$. In view of Sturm oscillation theorem, we have necessarily $z \geq z_{i+1}$, where $\lambda_i = z_i^2$ are the eigenvalues of Problem (1.1)-(1.2). Since $z_{i+1} \geq (i+1)\pi$, then

$$\frac{(i+1)\pi}{z} \leq 1, \quad i \geq 0. \quad (3.28)$$

Therefore, by (3.27) and (3.28),

$$\begin{aligned}
& \int_{\varphi^{-1}(i\pi + \frac{\pi}{2})}^{\varphi^{-1}(i\pi + \frac{3\pi}{2})} \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \\
& \geq \frac{\pi}{4} \frac{\frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2}}{1 - \frac{3q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{4z^2}} + \frac{\pi}{4} \log \left(1 - \frac{q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{z^2} \right) - \frac{\pi}{6} \log \left(1 - \frac{3q(\varphi^{-1}(i\pi + \frac{\pi}{2}))}{4z^2} \right)
\end{aligned}$$

$$-\frac{121\pi}{200} \frac{q'(\varphi^{-1}(c_1))}{z^2}. \quad (3.29)$$

On the other hand, it can be easily verified that $\frac{\log(1-s)-s}{(1-s)^2} \geq \frac{-1}{4}$ for $s \in [0, \frac{1}{11}]$. Using this, together with (3.18) and (3.28) (for $i = 0$), we obtain

$$\begin{aligned} & -2 \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{3\pi}{2})} \frac{q(x)}{z} \sin \varphi \cos \varphi \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \\ & \geq -\frac{\pi}{4} \left[\left(\frac{\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2}}{1-\frac{3}{4}\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2}} \right) \log \left(1 - \frac{3}{4} \frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2} \right) - 2 \left(\frac{\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2}}{1-\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2}} \right) \log \left(1 - \frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2} \right) \right] \\ & - \frac{11\pi}{80} \frac{q'(\varphi^{-1}(c_2))}{z^2}. \end{aligned} \quad (3.30)$$

Thus, Summing (3.29) and (3.30), one gets

$$\begin{aligned} & \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{3\pi}{2})} \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \\ & - 2 \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{3\pi}{2})} \frac{q(x)}{z} \sin \varphi \cos \varphi \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \\ & \geq \pi G \left(\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2} \right) - \frac{297\pi}{400z^2} q'(\varphi^{-1}(c)) \\ & \geq \pi G \left(\frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2} \right) - \frac{3\pi}{4z^2} q'(\varphi^{-1}(c)), \end{aligned} \quad (3.31)$$

where $G(s) = \frac{s}{4(1-\frac{3}{4}s)} \left(1 - \log(1 - \frac{3}{4}s) \right) + \frac{1}{4} \frac{(1+s)\log(1-s)}{4(1-s)} - \frac{1}{6} \log(1 - \frac{3}{4}s)$ and $q'(\varphi^{-1}(c)) = \max\{q'(\varphi^{-1}(c_1)), q'(\varphi^{-1}(c_2))\}$. It can be shown by straightforward computation that $G(s) \geq \frac{s}{10}$ for $s \in [0, \frac{1}{11}]$. By setting $s = \frac{q(\varphi^{-1}(i\pi+\frac{\pi}{2}))}{z^2} \geq \frac{q(0)}{z^2}$ and taking into account the condition $q'(x) \leq \frac{2}{15}q(0)$, we find out that

$$\begin{aligned} & \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2}+D)} \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \\ & - 2 \int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^{\varphi^{-1}(i\pi+\frac{\pi}{2})} \frac{q(x)}{z} \sin \varphi \cos \varphi \left(\int_{\varphi^{-1}(i\pi+\frac{\pi}{2})}^x \frac{q(s)}{z} \sin^2 \varphi ds \right) dx \\ & \geq 0. \end{aligned} \quad (3.32)$$

Now, according to Lemma 3.2 together with (3.32), we conclude that

$$\int_{\varphi^{-1}(k\pi+\frac{\pi}{2})}^{\varphi^{-1}(k\pi+\frac{\pi}{2}+D)} r^2(x) \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx \geq 0. \quad (3.33)$$

Therefore, by (3.24), (3.25), (3.26) and (3.33) we have $\dot{\theta}(x_0, z) \geq 0$ for $z^2 \geq 11q(x_0)$ and $q'(x) \leq \frac{2}{15}q(0)$. Obviously, if there is a $z \geq \sqrt{11q(x_0)}$ with $\dot{\theta}(x_0, z) = 0$, then in view of (3.24),

$$\int_0^{\varphi^{-1}(\frac{\pi}{2})} r^2(x) \frac{q(x)}{z} \left(\sin^2 \varphi - \varphi \sin \varphi \cos \varphi \right) dx = 0$$

which is not possible unless $q \equiv 0$ in $[0, x_0]$. The theorem is proved. \square

4 Proof of Theorem 2.1

Let the potential $q(x)$ be monotone increasing in $[0, x_0]$ and monotone decreasing in $[x_0, 1]$. We denote by $\tilde{q}(x)$ the reverse of the potential, i.e., $\tilde{q}(x) = q(1 - x)$. Then $\tilde{y}(x, z)$ is the solution of the initial value problem

$$\begin{cases} -\tilde{y}''(x, z) + \tilde{q}(x)\tilde{y}(x, z) = z^2\tilde{y}(x, z), \\ \tilde{y}(0) = 0, \quad \tilde{y}'(0) = 1. \end{cases} \quad (4.1)$$

As in section 2, we define the associated Prüfer transformation

$$\begin{cases} \tilde{y}(x, z) = \tilde{r}(x, z) \sin(z\tilde{\theta}(x, z)), \\ \tilde{y}'(x, z) = z\tilde{r}(x, z) \cos(z\tilde{\theta}(x, z)), \\ \tilde{\theta}(0, z) = 0. \end{cases} \quad (4.2)$$

As in [4], we have the following relations:

$$\begin{cases} \tilde{y}(x, z_n) = (-1)^{n+1} \frac{y(1-x, z_n)}{z_n r(1, z_n)}, \\ \tilde{r}(x, z_n) = \frac{r(1-x, z_n)}{r(1, z_n)}, \\ \tilde{\theta}(x, z_n) = \frac{n\pi}{z_n} - \theta(1-x, z_n), \end{cases} \quad (4.3)$$

where $\lambda_n = z_n^2$ is an eigenvalue of Problem (1.1)-(1.2).

Proof. of Theorem 2.1. As $\tilde{q}(x) = q(1 - x)$, then $\tilde{q}(x)$ be monotone increasing in $[0, 1 - x_0]$ and monotone decreasing in $[1 - x_0, 1]$. We have $\tilde{q}'(x) = -q'(1 - x) \leq \frac{2}{15}\tilde{q}(0) = \frac{2}{15}q(1)$, thus for $x \in [0, 1]$, $|q'(x)| \leq q^*$ where $q^* = \frac{2}{15} \inf\{q(0), q(1)\}$. Therefore by Theorem 3.1, $\tilde{\theta}(1 - x_0, z)$ is increasing for $z \geq \sqrt{11\tilde{q}(1 - x_0)} = \sqrt{11q(x_0)}$. Consequently, the function $\Psi(z) = \theta(x_0, z) + \tilde{\theta}(1 - x_0, z)$ is increasing for $z \geq \sqrt{11q(x_0)}$. By (4.3), $z_n \Psi(z_n) = n\pi$. Let m be an integer such that $m < n$ and $\lambda_m \geq 11q(x_0)$. Then

$$\Psi(z_n) = \frac{n\pi}{z_n} \geq \Psi(z_m) = \frac{m\pi}{z_m},$$

which implies that $\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}$. On the other hand, if $q(x_0) \leq \frac{\pi^2}{11}$, then

$$z_1 = \sqrt{\lambda_1} \geq \sqrt{\pi^2 + q^-} \geq \sqrt{11q(x_0)}$$

where $q^- = \inf_{x \in [0, 1]} q(x)$. Thus, in this case $\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}$ for all $n > m \geq 1$. If equality holds, then

$\Psi(z_n) = \Psi(z_m)$, so that $\dot{\Psi}(z) = 0$ for some $z > 0$. Hence $\dot{\theta}(x_0, z) = \dot{\tilde{\theta}}(1 - x_0, z) = 0$, and in view of Theorem 3.1, $q \equiv 0$ in $[0, x_0]$ and $\tilde{q} \equiv 0$ in $[0, 1 - x_0]$, i.e., $q \equiv 0$ in $[0, 1]$. \square

References

- [1] M. Ashbaugh and R. Benguria, *On the ratio of the first two eigenvalues of Schrödinger operators with positive potentials*, *Differential Equations and Mathematical Physics* (I. W. Knowles and Y. Saito, eds.), *Lecture Notes in Math.*, vol. 1285, Springer-Verlag, Berlin, 1987, pp. 16-25. MR 89a:35151.
- [2] M. S. Ashbaugh and R. D. Benguria, *Optimal bounds for ratios of eigenvalues of one-dimensional Schrödinger operators with Dirichlet boundary conditions and positive potentials*, *Comm. Math. Phys.*, 124 (1989), 403 – 415.
- [3] M. S. Ashbaugh and R. D. Benguria, *Optimal lower bound for the gap between the first two eigenvalues of one-dimensional Schrödinger operators with symmetric single-well potentials*, *Proc. Amer. Math. Soc.*, 105, (1989), 419 – 424.
- [4] M. Horváth and M. Kiss, *A bound for ratios of eigenvalues of Schrödinger operators with single-well potentials*, *Proc. Amer. Math. Soc.*, 134, (2005), 1425 – 1434.
- [5] Yu-Ling Huang and C. K. Law, *Eigenvalue ratios for the regular Sturm-Liouville system*, *Proc. Amer. Math. Soc.*, 124, (1996), 1427 – 1436.
- [6] C.K. Law and Yu-Ling Huang, *Eigenvalue ratios and eigenvalue gaps of Sturm-Liouville operators*, *Proceedings of the Royal Society of Edinburgh*, 128A, 337 – 347, 1998.
- [7] B.M. Levitan and I. S. Sargsyan, *Introduction to spectral theory: Selfadjoint Ordinary Differential Operators*, American Mathematical Society, *Translation of Mathematical Monographs*, Vol 39, (1975).